HAL open science

Stochastic diffeomorphisms and homogenization of multiple integrals

Antoine Gloria

▶ To cite this version:

Antoine Gloria. Stochastic diffeomorphisms and homogenization of multiple integrals. Applied Mathematics Research eXpress, 2008, 2008 (abn001), pp.1-23. 10.1093/amrx/abn001 . inria-00176568v2

HAL Id: inria-00176568

https://inria.hal.science/inria-00176568v2

Submitted on 5 Oct 2007

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

INSTITUT NATIONAL DE RECHERCHE EN INFORMATIQUE ET EN AUTOMATIQUE

Stochastic diffeomorphisms and homogenization of multiple integrals

A. Gloria

N° 6316

October, 2007

Thème NUM

Rapport de recherche

ISSN 0249-6399

Stochastic diffeomorphisms and homogenization of multiple integrals

A. Gloria *

Thème NUM — Systèmes numériques
Projet MICMAC

Rapport de recherche n° 6316 — October, 2007 — 21 pages

Abstract: In [1], Blanc, Le Bris and Lions have introduced the notion of stochastic diffeomorphism together with a variant of stochastic homogenization theory for linear and monotone elliptic operators. Their proofs rely on the ergodic theorem and on the analysis of the associated corrector equation. In the present article, we provide another proof of their results using the formalism of integral functionals. We also extend the analysis to cover the case of quasiconvex integrands.

Key-words: stochastic homogenization, multiple integrals, quasiconvexity, Γ-convergence

* CERMICS - ENPC & INRIA Paris-Rocquencourt, 6 et 8 avenue Blaise Pascal - Champs sur Marne, France ; Contact: antoine.gloria@inria.fr

Difféomorphismes stochastiques et homogénéisation des intégrales multiples

Résumé : Dans [1], Blanc, Le Bris et Lions ont introduit la notion de difféomorphisme stochastique ainsi qu'une variante de la théorie de l'homogénéisation stochastique pour les opérateurs elliptiques linéaires et monotones. Leurs techniques de preuve sont basées sur le théorème ergodique et l'étude du problème de correcteurs associé. Dans cet article, nous donnons une preuve différente de leur résultat en utilisant le formalisme des fonctionnelles intégrales. Nous démontrons ainsi un résultat similaire nouveau pour les intégrandes quasiconvexes.

Mots-clés : homogénéisation stochastique, intégrales multiples, quasiconvexité, Γ convergence

Introduction

In [1], Blanc, Le Bris and Lions have introduced the notion of stochastic diffeomorphism together with a variant of stochastic homogenization theory for linear and monotone elliptic operators. Their proofs rely on the ergodic theorem and on the analysis of the associated corrector equation by the means of Tartar's oscillating test functions or two-scale convergence. In [2], they draw the link between this stochastic variant of the homogenization theory and their previous work on stochastic lattices in [3, 4]. Using another classical approach to the homogenization theory, we give an alternative proof of (some of) their results, and extend them to the quasiconvex case. Our proof, which closely follows the one by Dal Maso and Modica in [5], is based on the compactness of a class of integral functionals with respect to Γ-convergence, on the ergodic subadditive theorem and on an argument related to the invariance of a thermodynamic limit with respect to properly invading domains. The latter argument is interesting since it illustrates how the assumptions made by Blanc, Le Bris and Lions allow to obtain such an invariance, which is typical to statistical mechanics approaches.

Correspondingly, Γ-convergence results for discrete energies on stochastic lattices have been announced in [6] and will be detailed in [7].

This article is organized as follows. In the first section, we recall the stochastic framework introduced in [1] together with the subadditive ergodic theorem [5, 8]. In Section 2, we recall some homogenization results in terms of Γ-convergence of integral functionals and state our main results. The last section is dedicated to their proofs.

Contents

1 Stochastic framework

Throughout the paper, we denote by $(\Omega, \mathcal{F}, \mathbb{P})$ a probability space. For any random variable $X \in L^1(\Omega, d\mathbb{P})$, we denote by $\mathbb{E}(X) = \int_\Omega X(\omega)d\mathbb{P}(\omega)$ its expectation. For $n \geq 1$, we also consider a translation group $\{\tau_k\}_{k\in\mathbb{Z}^n}$ that acts on the probability space. We assume that it preserves the probability measure $\mathbb{P}$ and that it is ergodic in the following sense: for all $B \in \mathcal{F}$,

$$(\tau_k(B) = B \quad \forall k \in \mathbb{Z}^n) \quad \Longrightarrow \quad (\mathbb{P}(B) = 0 \text{ or } 1). \tag{1}$$

A random variable $\rho \in L^1_{\mathrm{loc}}(\mathbb{R}^n, L^1(\Omega))$ is said to be stationary if for all $z \in \mathbb{Z}^n$, for almost every $x \in \mathbb{R}^n$ and almost surely,

$$\rho(x + z, \omega) = \rho(x, \tau_z\omega). \tag{2}$$

This type of stationarity is discrete since $z \in \mathbb{Z}^n$, and is related to the periodicity in law considered by Dal Maso and Modica in [5]. It differs from the more classical continuous stationarity setting of [9], which is also discussed in Subsection 2.4.

1.1 Stationary stochastic diffeormorphisms

Given a probability space, an ergodic translation group and the notion of stationarity recalled above, Blanc, Le Bris and Lions have introduced in [1] the notion of stationary stochastic diffeomorphism.

Definition 1 *An application $\Phi : \mathbb{R}^d \times \Omega \to \mathbb{R}^d$, which is continuous in the first variable and measurable in the second variable, is said to be a stationary stochastic diffeomorphism if*

- *for $\mathbb{P}$-almost all ω, $\Phi(\cdot, \omega)$ is a diffeormorphism from $\mathbb{R}^n$ onto itself,*
- *$\nabla\Phi$ is stationary in the sense of* (2),

if its Jacobian is uniformly bounded from below

$$\operatorname{Ess\,Inf}_{\omega\in\Omega\, x\in\mathbb{R}^n}[\det(\nabla\Phi(x, \omega))] \geq \nu > 0 \tag{3}$$

and if its gradient is uniformly bounded from above

$$\operatorname{Ess\,Sup}_{\omega\in\Omega\, x\in\mathbb{R}^n}(|\nabla\Phi(x, \omega)|) \leq M < \infty. \tag{4}$$

In order to prove homogenization results related to stationary stochastic diffeomorphisms, we will need the following ergodic theorems.

1.2 Ergodic theorems

The ergodic theorem can be written as follows

Theorem 1 *[10, 1] Let $F \in L^\infty(\mathbb{R}^d, L^1(\Omega))$ be a stationary random variable in the sense of* (2)*. For $k = (k_1, k_2, \dots, k_d) \in \mathbb{Z}^d$, let denote by $|k|_\infty = \sup_{1\le i\le d} |k_i|$. Then*

$$\frac{1}{(2N+1)^d} \sum_{|k|_\infty \le N} F(x, \tau_k\omega) \to_{N\to\infty} \mathbb{E}(F(x,\cdot)) \text{ in } L^\infty(\mathbb{R}^d), \text{ almost surely.}$$

As a consequence, we have

$$F\left(\frac{x}{\epsilon}, \omega\right) \rightharpoonup^*_{\epsilon\to 0} \mathbb{E}\left(\int_Q F(x,\cdot)dx\right) \text{ in } L^\infty(\mathbb{R}^d), \text{ almost surely.}$$

In the present work we will mainly focus on multiple integrals, which requires the use of another form of the ergodic theorem, namely the subadditive ergodic theorem. We recall here the version by Dal Maso and Modica, which is a corollary of the original result by Akcoglu and Krengel. Let $\mathcal{A}$ denote the open bounded subsets of $\mathbb{R}^n$. A set function $\mu : \mathcal{A} \to \mathbb{R}$ is said to be subadditive if

$$\mu(A) \le \sum_{k\in K} \mu(A_k)$$

for every $A \in \mathcal{A}$ and for every finite family $(A_k)_{k\in K}$ in $\mathcal{A}$ such that

$$A_k \subset A \;\forall k \in K, \quad A_h \cap A_k = \emptyset \;\forall h, k \in K, h \ne h, \quad |A - \cup_{k\in K} A_k| = 0.$$

Let $\mathcal{M} = \mathcal{M}(c)$ be the family of the subadditive functions $\mu : \mathcal{A} \to \mathbb{R}$ such that

$$0 \le \mu(A) \le c|A| \qquad \forall A \in \mathcal{A}, \tag{5}$$

where c is a fixed positive constant.

Given the above definitions, there holds

Theorem 2 *[8, 10, 5] Let $\mu : \Omega \to \mathcal{M}$ be a subadditive process satisfying (5). If μ is stationary in the sense of (2), that is*

$$\forall A \in \mathcal{A},\ \forall z \in \mathbb{Z}^n,\ \tau_z\mu(\omega)(A) = \mu(\omega)(A+z) = \mu(\tau_z\omega)(A) \text{ almost surely,}$$

then μ is ergodic and there exists $\phi \in \mathbb{R}$, such that for $\mathbb{P}$-almost every ω and for every cube $\tilde{Q}$ in $\mathbb{R}^n$,

$$\lim_{t\to\infty} \frac{\mu(\omega)(t\tilde{Q})}{|t\tilde{Q}|} = \phi.$$

2 Main results

In the first subsection, we recall some definitions and properties of integral functionals. The main results are presented in the three following subsections. First, we give a variant starting from the periodic case, then we provide a variant starting from the classical stochastic case.

2.1 Homogenization and Γ-convergence

We assume the reader is familiar with the basic properties of the Γ-convergence theory. Should the need arise, [11, 12] provides with a good introduction and [13] gives a more systematic study of the subject. For consistency, let us recall some notation and properties of the Γ-convergence in Sobolev spaces. In the sequel, $W^{1,p}(A)$ denotes the Sobolev space for $A \in \mathcal{A}$ and $p > 1$.

Definition 2 *Let $A \in \mathcal{A}$ and $I_\epsilon : L^p(A) \to \mathbb{R}$ be a family of functionals. We say that I_ϵ $\Gamma(L^p)$-converges to $I : L^p(\mathbb{R}^n) \to \mathbb{R}$ on Ω if and only if the two following properties are satisfied.*
(i) Liminf inequality: for every $u \in L^p(\Omega)$ and every sequence u_ϵ such that $u_\epsilon \to u$ in $L^p(\Omega)$,

$$I(u) \leq \liminf_{\epsilon} I_\epsilon(u_\epsilon).$$

(ii) Recovery sequence: for every $u \in L^p(\Omega)$ there exists a sequence $\bar{u}_\epsilon$ such that $\bar{u}_\epsilon \to u$ in $L^p(\Omega)$ and

$$\limsup_{\epsilon} I_\epsilon(\bar{u}_\epsilon) \leq I(u).$$

Definition 2 is also refered to as the sequential Γ-convergence since it is stated using the convergence of sequences. We refer to [11, Section 1.4] for other definitions, which are equivalent in the present context.

The Γ-convergence implies the convergence of minima and minimizers of functions as stated in the following

Lemma 1 *Let $A \in \mathcal{A}$ and $I_\epsilon : L^p(A) \to \mathbb{R}$ be a family of functions that $\Gamma(L^p)$-converges to I on A. If I_ϵ is lower semicontinuous for the weak topology of $W^{1,p}(A)$ and equicoercive in the following sense*

$$\exists \quad c > 0, \quad \forall v \in W^{1,p}(A), \forall \epsilon > 0, \quad c\|\nabla v\|^p_{L^p(A)} \leq I_\epsilon(v)$$

then for every $u_0 \in W^{1,p}(A)$

$$\lim_{\epsilon \to 0} \Big(\inf\{I_\epsilon(v + u_0), v \in W^{1,p}_0(A)\} \Big) = \inf\{I(v + u_0), v \in W^{1,p}_0(A)\}$$

and for every sequence u_ϵ of minimizers of $\inf\{I_\epsilon(v + u_0), v \in W^{1,p}_0(A)\}$ there exists a subsequence (not relabeled) and a minimizer u of $\inf\{I(v + u_0), v \in W^{1,p}_0(A)\}$ such that $u_\epsilon \rightharpoonup u$ in $W^{1,p}(A)$.

In order to recall the homogenization theorems, let us introduce some further definitions.

Definition 3 *A function $W : \mathbb{R}^n \times \mathbb{R}^{n\times d} \to \mathbb{R}$ is a Carathéodory function if for every $\xi \in \mathbb{R}^{n\times d}$, $W(\cdot, \xi)$ is measurable and if for almost all $x \in \mathbb{R}^n$, $W(x, \cdot)$ is continuous.*

A Carathéodory function satisfies a standard growth condition of order $p > 1$ if there exist $C \geq c > 0$ such that for almost all $x \in \mathbb{R}^n$ and for all $\xi \in \mathbb{R}^{n\times d}$,

$$c|\xi^p| - 1 \leq W(x,\xi) \leq C(1+|\xi|^p). \tag{6}$$

Definition 4 *A Carathéodory function $W : \mathbb{R}^n \times \mathbb{R}^{n\times d} \to \mathbb{R}$ is ($W^{1,p}$-)quasiconvex if and only if, for almost every $x \in \mathbb{R}^n$ and for all $\xi \in \mathbb{R}^{n\times d}$,*

$$W(x,\xi) = \inf\left\{\int_{(0,1)^n} W(x,\xi+\nabla_y v(y))dy, v \in W^{1,p}((0,1)^n,\mathbb{R}^d)\right\}.$$

A Carathéodory function on $\mathbb{R}^n \times \mathbb{R}^{n\times d}$ is equivalent to a Borel function on $\mathbb{R}^n \times \mathbb{R}^{n\times d}$.

We are now in position to define the class of energy densities we will deal with throughout the article.

Definition 5 *A function $W : \mathbb{R}^n \times \mathbb{R}^{n\times d} \to \mathbb{R}$ is a standard energy density if*

- *W is a Carathéodory function,*
- *W is quasiconvex,*
- *W satisfies the p-growth condition (6).*

The functionals we consider are then of the form

$$\begin{array}{cccl} I: & L^p(A) & \to & \mathbb{R} \\ & u & \mapsto & \begin{cases} \displaystyle\int_A W(x,\nabla u) & \text{if } u \in W^{1,p}(A), \\ \infty & \text{else} \end{cases} \end{array}$$

where W is a standard energy density. Such integral functionals are lower-semicontinuous for the weak topology of $W^{1,p}(A)$, which allows us to use the direct method of the calculus of variations to prove that minimum problems admit solutions (recall that such functionals are coercive).

In the following, Γ denotes the $\Gamma(L^p)$-convergence, $\mathcal{L}^k$ stands for the k-dimensional Lebesgue measure and $Q_N = (0,N)^n$ ($Q = (0,1)^n$). We are now in position to state the main results of the paper, that are generalizations of the periodic and stationary stochastic cases in the sense by Blanc, Le Bris and Lions.

2.2 The periodic variant

A first variant consists in considering a periodic energy density and modify it using a stochastic diffeormorphism, that is considering $W(\Phi^{-1}(\frac{x}{\epsilon},\omega),\xi)$ instead of $W(\frac{x}{\epsilon},\xi)$ as it is usually done in periodic homogenization. The result is the following.

Hypotheses 1 $W : \mathbb{R}^n \times \mathbb{R}^{n\times d} \to \mathbb{R}^+$ *satisfies*

- *W is a standard energy density of order $p > 1$,*
- *W is Q-periodic: $W(x + e, \xi) = W(x, \xi)$ for almost every $x \in \mathbb{R}^n$ and for all $e \in \mathbb{Z}^n$ and $\xi \in \mathbb{R}^{n\times d}$.*

Theorem 3 *Let Φ be a stationary stochastic diffeomorphism and let W satisfy Hypotheses 1. For all $A \in \mathcal{A}$ and for $\epsilon > 0$, let denote by*

$$\begin{array}{rccl} I_\epsilon(\cdot,\omega): & L^p(A) & \to & \mathbb{R}, \\ & u & \mapsto & \left\{ \begin{array}{ll} \displaystyle\int_A W\left(\Phi^{-1}\left(\frac{x}{\epsilon},\omega\right), \nabla u\right) & \textit{if } u \in W^{1,p}(A), \\ \infty & \textit{else} \end{array} \right. \end{array} \tag{7}$$

For $\mathbb{P}$-almost every ω the integral functionals $I_\epsilon(\cdot,\omega)$ $\Gamma(L^p)$-converge as $\epsilon \to 0$ to the deterministic integral functional

$$\begin{array}{rccl} I^*: & L^p(A) & \to & \mathbb{R}, \\ & u & \mapsto & \left\{ \begin{array}{ll} \displaystyle\int_A W^*(\nabla u) & \textit{if } u \in W^{1,p}(A), \\ \infty & \textit{else} \end{array} \right. \end{array} \tag{8}$$

where W^* is a standard energy density satisfying the following asymptotic formulas for all $\xi \in \mathbb{R}^{n\times d}$

$$\begin{aligned} W^*(\xi) &= \mathbb{E}\left(\lim_{N\to\infty} \frac{1}{N^d} \inf\left\{\int_{Q_N} W(\Phi^{-1}(y,\cdot), \xi + \nabla v)dy, v \in W_0^{1,p}(Q_N), \mathbb{R}^d)\right\}\right) \\ &= \mathbb{E}\left(\lim_{N\to\infty} \frac{1}{N^d} \inf\left\{\int_{\Phi(Q_N,\cdot)} W(\Phi^{-1}(y,\cdot), \xi + \nabla v)dy, v \in W_0^{1,p}(\Phi(Q_N,\cdot), \mathbb{R}^d)\right\}\right) \\ &\qquad \times \det\left(\mathbb{E}\left(\int_Q \nabla\Phi(z,\cdot)dz\right)\right)^{-1} \end{aligned} \tag{9}$$

Due to Lemma 1, Theorem 3 also implies the convergence of minimum problems on Sobolev spaces. In the particular case of strictly convex energies, we recover the homogenization property of the associated monotone system of elliptic equations (that is the Euler-Lagrange equations of the minimization problem) dealt with in [1].

For $\Phi(x,\cdot) = x$, we recover the classical result by Braides (see [14, 15]) and the classical asymptotic formula for periodic quasiconvex integrands.

In the case for which $\nabla\Phi(\cdot,\omega)$ is kQ-periodic with $k \in \mathbb{Z}^d$, it makes sense to replace the test space $W_0^{1,p}(\Phi(Q_N,\cdot), \mathbb{R}^d)$ by the space of periodic functions $W_\#^{1,p}(\Phi(Q_N,\cdot), \mathbb{R}^d)$ in (9). If the energy density is strictly convex, then the limit is attained for "k cells" by the uniqueness of the minimizer (see [15, Section 14.3] for details). Otherwise, the counter-example due to Müller in [16] shows that one single periodic pattern is not enough.

2.3 The stochastic variant

For the second variant, we start from another standard assumption of the homogenization theory: the classical stationary ergodic case dealt with by Dal Maso and Modica in [5]. Instead of considering $W(\frac{x}{\epsilon},\omega,\xi)$, we consider the energy density "deformed by" a stochastic diffeomorphism: $W(\Phi^{-1}(\frac{x}{\epsilon},\omega),\omega,\xi)$. Since Φ^{-1} is not necessarily stationary for the action group, the result by Dal Maso and Modica does not cover this case. Let us detail the assumptions on the energy density.

Hypotheses 2 $W:\mathbb{R}^n\times\Omega\times\mathbb{R}^{n\times d}\to\mathbb{R}^+$ *satisfies*

- *W is measurable on $\mathbb{R}^n\times\Omega\times\mathbb{R}^{n\times d}$,*
- *$W(\cdot,\omega,\cdot)$ is a standard energy density of order $p>1$ satisfying (6) almost surely,*
- *for almost every $x\in\mathbb{R}^n$ and for all $\xi\in\mathbb{R}^{n\times d}$, $W(x+z,\omega,\xi)=W(x,\tau_z\omega,\xi)$ for all $z\in\mathbb{Z}^n$.*

Theorem 4 *Let Φ be a stationary stochastic diffeomorphism and W satisfy Hypotheses 2. For all $A\in\mathcal{A}$ and for $\epsilon>0$, let denote by*

$$\begin{array}{rcl} I_\epsilon(\cdot,\omega): \quad L^p(A) & \to & \mathbb{R}, \\ u & \mapsto & \begin{cases} \displaystyle\int_A W\left(\Phi^{-1}\left(\frac{x}{\epsilon},\omega\right),\omega,\nabla u\right) & \text{if } u\in W^{1,p}(A), \\ \infty & \text{else} \end{cases} \end{array} \tag{10}$$

For $\mathbb{P}$-almost every ω, the functionals I_ϵ $\Gamma(L^p)$-converge as $\epsilon\to 0$ to the deterministic integral functional

$$\begin{array}{rcl} I^*: \quad L^p(A) & \to & \mathbb{R}, \\ u & \mapsto & \begin{cases} \displaystyle\int_A W^*(\nabla u) & \text{if } u\in W^{1,p}(A), \\ \infty & \text{else} \end{cases} \end{array} \tag{11}$$

where W^* is a standard energy density satisfying the following asymptotic formulas for all $\xi\in\mathbb{R}^{n\times d}$

$$\begin{aligned} W^*(\xi) &= \mathbb{E}\left(\lim_{N\to\infty}\frac{1}{N^d}\inf\left\{\int_{Q_N} W(\Phi^{-1}(y,\cdot),\cdot,\xi+\nabla v)dy, v\in W_0^{1,p}(Q_N),\mathbb{R}^d)\right\}\right) \\ &= \mathbb{E}\left(\lim_{N\to\infty}\frac{1}{N^d}\inf\left\{\int_{\Phi(Q_N,\cdot)} W(\Phi^{-1}(y,\cdot),\cdot,\xi+\nabla v)dy, v\in W_0^{1,p}(\Phi(Q_N,\cdot),\mathbb{R}^d)\right\}\right) \\ &\qquad\qquad\qquad \times\det\left(\mathbb{E}\left(\int_Q \nabla\Phi(z,\cdot)dz\right)\right)^{-1} \end{aligned}$$

As for the periodic case, Theorem 4 implies the convergence of infimum problems on Sobolev spaces and can be recast in terms of monotone elliptic systems in the strictly convex case.

For $\Phi(x,\cdot) = x$, we recover the classical result by Dal Maso and Modica for convex integrands in [5].

2.4 Remarks and extensions

As pointed out by Blanc, Le Bris and Lions in [1], another variant of the above results can be obtained by using another ergodic translation group. Let us consider the translation group $\{\tau_z\}_{z\in\mathbb{R}^n}$ and replace (1) by

$$(\tau_z(B) = B \quad \forall z \in \mathbb{R}^n) \quad \Longrightarrow \quad (\mathbb{P}(B) = 0 \text{ or } 1). \tag{12}$$

Then, up to considering a continuous stationarity that holds for all $z \in \mathbb{R}^n$ in Hypotheses 2 and for the stochastic diffeomorphism, Theorem 4 still holds.

As will be made clear in Section 3, the proofs do not depend on the type of stationarity considered provided Theorem 2 holds, which is also the case with an ergodic continuous translation group, as briefly recalled in Subsection 3.4.

A heuristic argument developed later in Remark 2 allows us to identify sufficient properties on the stochastic diffeomorphisms and energy densities in order for the homogenization result to hold. The following lemma gives a particular case for which the argument applies.

Lemma 2 *Let $W(x,\omega,\xi)$ be a stochastic family of standard energy densities and Φ be a stationary stochastic diffeomorphism, and denote by*

$$\mathcal{V} = \det\left(\mathbb{E}(\int_Q \nabla\Phi(z,\cdot)dz)\right)^{-1}.$$

For all $A \in \mathcal{A}$, $\mathbb{P}$-almost surely, $\lim_{\epsilon\to 0} \frac{1}{|\frac{1}{\epsilon}A|}\int_{\Phi^{-1}(\frac{1}{\epsilon}|A|,\omega)} dx = \mathcal{V}$ ([1, Lemme 2.1]). If there exists a function $\phi^* : \mathbb{R}^{n\times d} \to \mathbb{R}$ such that for all $A \in \mathcal{A}$ the functional

$$\begin{array}{rrcl} J_\epsilon(\cdot,\omega): & L^p(A) & \to & \mathbb{R}, \\ & u & \mapsto & \left\{ \begin{array}{ll} \displaystyle\int_A W\Big(y,\omega,\left(\nabla\Phi\left(y,\omega\right)\right)^{-1}\nabla u\Big)\det(\nabla\Phi(y,\omega)) & \textit{if } u \in W^{1,p}(A), \\ \infty & \textit{else} \end{array}\right. \end{array} \tag{13}$$

Γ-converges $\mathbb{P}$-almost everywhere to the functional

$$\begin{array}{rrcl} J^*: & L^p(A) & \to & \mathbb{R}, \\ & u & \mapsto & \left\{ \begin{array}{ll} \displaystyle\int_A \phi^*(\nabla u) & \textit{if } u \in W^{1,p}(A), \\ \infty & \textit{else} \end{array}\right. \end{array} \tag{14}$$

then, the functional

$$I_\epsilon(\cdot,\omega): \quad L^p(A) \to \mathbb{R}, \qquad u \mapsto \begin{cases} \displaystyle\int_A W\left(\Phi^{-1}\left(\frac{x}{\epsilon},\omega\right),\omega,\nabla u\right) & \textit{if } u \in W^{1,p}(A), \\ \infty & \textit{else} \end{cases} \tag{15}$$

Γ*-converges to* $I^* = \mathcal{V}J^*$*.*

Subsection 3.3 is dedicated to the proof of Lemma 2. As discussed in [2], we may think of mixing two types of stationarity, one for W and another one for $\nabla\Phi$. A simple example is provided by the so-called almost periodic case for which W may be 1-periodic and $\nabla\Phi$ α-periodic with $\alpha \notin \mathbb{Q}$. In this case, it is well-known that the homogenization property holds for J_ϵ (see [14] or [15, Theorem 15.3]), so does it for I_ϵ due to Lemma 2. For more complex cases, the Γ-convergence of J_ϵ is unclear.

It should also be noticed that the main ingredients of our proofs (the compactness of Γ-convergence and the subadditive ergodic theorem for sublinear processes) hold true in many cases that are not detailed in the present work. Similar generalizations of the results derived in [17] can be obtained using the very same arguments.

3 Proofs of the main results

The proofs of the main results follow the approach by Marcellini in [18], Braides in [14, 15], and Dal Maso and Modica in [5]. As a first step, we recall a classical compactness result for a family of integral functionals uniformly satisfying a standard growth condition of order $p > 1$. Doing so, for ω, up to extraction, we obtain an integral representation formula for the limit. The second step consists in proving that the associated integrand $\widetilde{W}(x,\omega,\cdot)$ does not depend on ω and x almost surely. This is done using the subadditive ergodic theorem and the convergence of minimum problems. We then conclude the proof using the uniqueness of the limit.

The first two subsections are dedicated to the proof of Theorems 3 and 4. In the last subsection, we briefly explain why Theorem 2 still holds in the setting of an ergodic continuous translation group.

3.1 Compactness result

Let us recall the compactness result for standard integral functionals with respect to Γ-convergence.

Theorem 5 *[15, Theorem 12.25] Let* (W_ϵ) *be a family of standard energy densities uniformly satisfying* (6)*. Then, for every sequence* (ϵ_j) *of positive real numbers converging to*

0, there exists a subsequence (ϵ_{j_k}) *and a standard energy density* $\tilde{W}$*, such that the integral functionals*

$$\begin{array}{rccl} J_\epsilon: & L^p(A) & \to & \mathbb{R}, \\ & u & \mapsto & \left\{ \begin{array}{ll} \displaystyle\int_A W_\epsilon(x,\nabla u) & \textit{if } u \in W^{1,p}(A), \\ \infty & \textit{else} \end{array} \right. \end{array} \qquad (16)$$

Γ*-converge to the functional*

$$\begin{array}{rccl} J: & L^p(A) & \to & \mathbb{R}, \\ & u & \mapsto & \left\{ \begin{array}{ll} \displaystyle\int_A \tilde{W}(x,\nabla u) & \textit{if } u \in W^{1,p}(A) \\ \infty & \textit{else} \end{array} \right. \end{array} \qquad (17)$$

for all $A \in \mathcal{A}$*.*

For $\mathbb{P}$-almost ω, $(x,\xi) \mapsto W(\Psi^{-1}(\frac{x}{\epsilon},\omega),\xi)$ is such a family of stantard integrands. Thus, for every fixed $\omega \in \Omega$, up to extraction, the associated integral functionals Γ-converge to a standard integral functional, whose integrand will be denoted by $(x,\omega,\xi) \mapsto \tilde{W}(x,\omega,\xi)$. Theorem 3 is proved if the asymptotic formula (9) exists and if $\tilde{W}(x,\omega,\xi) = W^*(\xi)$ for almost all x and all $\xi \in \mathbb{R}^{n\times d}$ almost surely.

3.2 Existence of the asymptotic limit for the periodic variant

By the following change of variable: $y = \Phi^{-1}(\frac{x}{\epsilon},\omega)$, we have

$$\begin{array}{rcl} I_\epsilon(u,\omega) & = & \displaystyle\int_A W\left(\Phi^{-1}\left(\frac{x}{\epsilon},\omega\right),\nabla u\right)dx \\ & = & \displaystyle\epsilon^n \int_{\Phi^{-1}\left(\frac{1}{\epsilon}A,\omega\right)} W\left(y,\left(\nabla\Phi(y,\omega)\right)^{-1}\nabla\tilde{u}\right)\det\left(\nabla\Phi(y,\omega)\right)dy, \end{array} \qquad (18)$$

where $\tilde{u}(y) = \frac{1}{\epsilon}u(\epsilon\Phi(y,\omega))$. It is worth noticing that

$$\forall\epsilon > 0 \text{ and almost surely,} \quad (\, u \in W_0^{1,p}(A,\mathbb{R}^d)\,) \quad \Longleftrightarrow \quad (\, \tilde{u} \in W_0^{1,p}\left(\Phi^{-1}\left(\frac{1}{\epsilon}A,\omega\right),\mathbb{R}^d\right)\,).$$

In what follows, for all $\xi \in \mathbb{R}^{n\times d}$, l_ξ denotes $\mathbb{R}^n \to \mathbb{R}^d$, $x \mapsto \xi\cdot x$, so that $\nabla l_\xi = \xi$. We then denote by $\tilde{l}_\xi$ the function $\mathbb{R}^n \to \mathbb{R}^d$, $y \mapsto l_\xi(\Phi(y,\omega))$, so that $\nabla\tilde{l}_\xi = \nabla\Phi(y,\omega)\,\xi$.

Let us now consider the simpler related problem: the convergence properties of the functional

$$G : (A, v_A, \omega) \mapsto \int_A W\left(y,\left(\nabla\Phi(y,\omega)\right)^{-1}\nabla v_A\right)\det\left(\nabla\Phi(y,\omega)\right)dy,$$

where $A \in \mathcal{A}$, $v_A \in W^{1,p}(A)$ and $\omega \in \Omega$. Let us point out that

$$G(A, \nabla v_A, \omega) = \int_{\Phi(A,\omega)} W\left(\Phi^{-1}(y,\omega), \nabla u\right) dy.$$

We define a subadditive process as follows. For any fixed $\xi \in \mathbb{R}^{n\times d}$, let

$$\mu_\xi : \omega \mapsto \mu_\xi(\omega),$$

where

$$\begin{aligned} \mu_\xi(\omega): \quad & \mathcal{A} \to \mathbb{R} \\ & A \mapsto \inf\left\{G(A, \tilde{l}_\xi + v, \omega), v \in W_0^{1,p}(A, \mathbb{R}^n)\right\}. \end{aligned}$$

Let us prove that $\mu_\xi \in \mathcal{M}$. The subadditivity of μ_ξ is a consequence of the subadditivity of integrals, noticing that for all $A, B \in \mathcal{A}$ such that $A \cap B = \emptyset$,

$$u_A \in W_0^{1,p}(A, \mathbb{R}^n), u_B \in W_0^{1,p}(B, \mathbb{R}^n) \quad \Longrightarrow \quad u_A 1_A + u_B 1_B \in W_0^{1,p}(A \cup B, \mathbb{R}^n),$$

where 1_A and 1_B denote the characteristic functions of the sets A and B. It remains to prove that μ_ξ is sublinear. Due to (4), which gives a uniform bound on the determinant, and to the growth condition (6), the inequality (5) is proved if $(\nabla\Phi)^{-1}$ is uniformly bounded. This is a consequence of the definition of a stochastic diffeomorphism: due to (3) and (4),

$$\|(\nabla\Phi)^{-1}\|_{L^\infty} = \left\|\frac{1}{\det(\nabla\Phi)} \mathrm{cof}\,(\nabla\Phi)\right\|_{L^\infty} \leq \frac{(n-1)!}{\nu} M^n. \tag{19}$$

The integrand of G is stationary in the sense of (2), thus it is ergodic, which implies the ergodicity of the subadditive process μ_ξ according to [5]. Thus, applying Theorem 2, there exists $\phi(\xi) \in \mathbb{R}$, such that for every cube $\tilde{Q}$ in $\mathbb{R}^d$ and $\mathbb{P}$-almost every $\omega \in \Omega$,

$$\lim_{t\to\infty} \frac{\mu_\xi(\omega)(t\tilde{Q})}{|t\tilde{Q}|} = \phi(\xi).$$

Let us prove now that the result holds for any family of domains invading properly $\mathbb{R}^d$, in the spirit of the existence of thermodynamic limits in statistical mechanics (see [19]).

Let us recall the most important property of stationary stochastic diffeomorphisms in the present context.

Lemma 3 *[1, Lemme 2.1] Let Φ be a stationary stochastic diffeomorphism. Then*

$$\nabla\Phi\left(\frac{x}{\epsilon}, \omega\right) \rightharpoonup^* \mathbb{E}\left(\int_Q \nabla\Phi(z,\cdot)dz\right) \text{ in } (L^\infty(\mathbb{R}^n))^n \quad \text{almost surely.}$$

Therefore, almost surely,

$$\epsilon_n \Phi\left(\frac{\cdot}{\epsilon_n}, \omega\right) \to L(\cdot) \text{ strongly in } L^\infty_{loc}(\mathbb{R}^n) \tag{20}$$

where $L : \mathbb{R}^n \ni x \mapsto \mathbb{E}(\int_Q \nabla\Phi(z,\cdot)dz) \cdot x$.

Due to (19), Φ^{-1} is a C-Lipschitz function, with $C>0$. Therefore

$$\begin{array}{rl} & |\epsilon\Phi^{-1}(\frac{L(x)}{\epsilon},\omega)-\epsilon\Phi^{-1}(\Phi(\frac{x}{\epsilon},\omega),\omega)| \\ \leq & \epsilon C|\frac{1}{\epsilon}L(x)-\Phi(\frac{x}{\epsilon},\omega)| \\ \leq & C|L(x)-\epsilon\Phi(\frac{x}{\epsilon},\omega)|, \end{array}$$

and Lemma 3 implies

$$\epsilon\Phi^{-1}(\frac{\cdot}{\epsilon},\omega)\to L^{-1}(\cdot) \text{ strongly in } L^\infty_{loc}(\mathbb{R}^n). \tag{21}$$

This tells us that $\Phi^{-1}(tQ,\omega)$ is "close to" $L^{-1}(tQ)$.

Let us first study the limit on $L^{-1}(tQ)$ by considering the process:

$$\nu_{L,\xi}:\omega\mapsto\nu_{L,\xi}(\omega),$$

where

$$\begin{array}{rl} \nu_{L,\xi}(\omega): & \mathcal{A}\to\mathbb{R} \\ & A\mapsto\inf\Big\{G(L^{-1}(A),\tilde{l}_\xi+v,\omega),v\in W^{1,p}_0(L^{-1}(A),\mathbb{R}^n)\Big\}. \end{array}$$

We shall prove that for all cube $\tilde{Q}$ in $\mathbb{R}^n$

$$\lim_{t\to\infty}\frac{\nu_{L,\xi}(\omega)(t\tilde{Q})}{|t\tilde{Q}|}=\mathcal{V}\phi(\xi), \tag{22}$$

where $\mathcal{V}=\det L^{-1}=\det\Big(\mathbb{E}(\int_Q\nabla\Phi(z,\cdot)dz)\Big)^{-1}$.

Let us first prove the inequality

$$\limsup_{t\to\infty}\frac{\nu_{L,\xi}(\omega)(t\tilde{Q})}{|t\tilde{Q}|}\leq\mathcal{V}\phi(\xi). \tag{23}$$

For all $\eta>0$, there exists a set of disjoint cubes $Q_{i=1,\dots,K}\subset\tilde{Q}$ such that $|\tilde{Q}\setminus\cup_{i=1,K}Q_i|\leq\eta$. Constructing a test function on $W^{1,p}_0(t\tilde{Q})$ starting from tests functions on $W^{1,p}_0(tQ_i)$, extended by 0 on $t\tilde{Q}\setminus\cup_{i=1,K}tQ_i$, we obtain

$$\begin{array}{rl} \dfrac{\nu_{L,\xi}(\omega)(t\tilde{Q})}{|t\tilde{Q}|}\leq & \dfrac{1}{|t\tilde{Q}|}\left(\displaystyle\sum_{i=1,K}\mu_\xi(\omega)(tQ_i)\right)+\eta C(1+|\xi|^p) \\ \leq & \dfrac{|L^{-1}(\tilde{Q})|}{|\tilde{Q}|}\left(\displaystyle\sum_{i=1,K}\frac{|Q_i|}{|L^{-1}(\tilde{Q})|}\frac{1}{|tQ_i|}\mu_\xi(\omega)(tQ_i)\right)+\eta C(1+|\xi|^p). \end{array}$$

Taking first the limsup $t\to\infty$ and then the limit $\eta\to0$, we obtain the desired inequality almost surely.

To prove the converse inequality, let us consider $\overline{Q}$, the smaller cube containing $L^{-1}(\tilde{Q})$. For all $\eta > 0$, there exists a set of disjoint cubes $Q_{i=1,\ldots,K} \subset \overline{Q} \setminus L^{-1}(\tilde{Q})$ such that $|\overline{Q} \setminus (L^{-1}(\tilde{Q}) \cup_{i=1,K} Q_i)| \leq \eta$. Let denote by $\chi = \dfrac{|L^{-1}(\tilde{Q})|}{|\overline{Q}|} \leq 1$. Proceeding as above, one has

$$\begin{aligned}
\frac{\mu_\xi(\omega)(t\overline{Q})}{|t\overline{Q}|} \leq \quad & \frac{\nu_{L,\xi}(\omega)(t\tilde{Q})}{|t\overline{Q}|} + \frac{1}{|t\overline{Q}|}\left(\sum_{i=1,K} \mu_\xi(\omega)(tQ_i)\right) + \eta C(1+|\xi|^p) \\
\leq \quad & \frac{\chi}{\mathcal{V}} \frac{\nu_{L,\xi}(\omega)(t\tilde{Q})}{|t\tilde{Q}|} + \left(\sum_{i=1,K} \frac{|Q_i|}{|\overline{Q}|} \frac{1}{|tQ_i|} \mu_\xi(\omega)(tQ_i)\right) + \eta C(1+|\xi|^p).
\end{aligned}$$

Taking first the liminf $t \to \infty$, we obtain

$$\phi(\xi) \leq \frac{\chi}{\mathcal{V}} \liminf_{t\to\infty} \frac{\nu_{L,\xi}(\omega)(t\tilde{Q})}{|t\tilde{Q}|} + \phi(\xi)\frac{\sum_{i=1,K}|Q_i|}{|\overline{Q}|} + \eta C(1+|\xi|^p).$$

Since $\sum_{i=1,K}|Q_i| = |\overline{Q}| - |L^{-1}(\tilde{Q})| - \eta = |\overline{Q}|(1-\chi) - \eta$, we deduce

$$\mathcal{V}\phi(x) \leq \liminf_{t\to\infty} \frac{\nu_{L,\xi}(\omega)(t\tilde{Q})}{|t\tilde{Q}|}. \tag{24}$$

Combining (23) and (24), we obtain (22).

Let us now deal with $\Phi^{-1}(tQ, \omega')$ where $\omega' \in \Omega$ is fixed. We claim that, almost surely, for any $B \in \mathcal{A}$ such that $B \subset\subset L^{-1}(\tilde{Q})$, there exists $M \in \mathbb{N}$ such that $B \subset \epsilon\Phi^{-1}(\frac{\tilde{Q}}{\epsilon}, \omega')$ for all $\epsilon \leq 1/M$. We denote by $\eta = d(\partial B, \partial L^{-1}(\tilde{Q})) > 0$. Due to the uniform convergence (21), there exists $\alpha : \mathbb{R}^+ \to \mathbb{R}^+$ such that for all $y \in \tilde{Q}$,

$$|\epsilon\Phi^{-1}(\frac{y}{\epsilon}, \omega') - L^{-1}(y)| \leq \alpha(\epsilon),$$

and $\lim_{\epsilon\to 0}\alpha(\epsilon) = 0$. Therefore, $d(\epsilon\Phi^{-1}(\frac{\partial\tilde{Q}}{\epsilon}, \omega'), \partial L^{-1}(\tilde{Q})) \leq \alpha(\epsilon)$. Since the diffeomorphism preserves orientation, $\{y \in L^{-1}(\tilde{Q}), d(y, \partial L^{-1}(\tilde{Q})) \geq \alpha(\epsilon)\} \subset \epsilon\Phi^{-1}(\frac{\tilde{Q}}{\epsilon}, \omega')$. Thus, for ϵ small enough, $B \subset \epsilon\Phi^{-1}(\frac{\tilde{Q}}{\epsilon}, \omega')$, and

$$|\epsilon\Phi^{-1}(\frac{\tilde{Q}}{\epsilon}, \omega') \setminus B| \leq 2\eta M^n \frac{(n-1)!}{\nu}|\partial\tilde{Q}|. \tag{25}$$

Correspondingly, for any $D \in \mathcal{A}$ such that $L^{-1}(\tilde{Q}) \subset\subset D$, there exists $M \in \mathbb{N}$ such that $\epsilon\Phi^{-1}(\frac{\tilde{Q}}{\epsilon}, \omega') \subset D$ for $\epsilon \leq 1/M$. An estimate similar to (25) also holds.

In other words, for all cube $\overline{Q} \subset \tilde{Q}$, $tL^{-1}(\overline{Q}) \subset \Phi^{-1}(t\tilde{Q}, \omega')$ for t big enough.

Let

$$\nu_{\Phi(\omega'),\xi} : \omega \mapsto \nu_{\Phi(\omega'),\xi}(\omega)$$

denote the process

$$\begin{aligned}\nu_{\Phi(\omega'),\xi}(\omega): \quad & \mathcal{A} \to \mathbb{R} \\ & A \mapsto \inf\left\{G(\Phi^{-1}(A,\omega'), \tilde{l}_\xi + v, \omega), v \in W_0^{1,p}(L^{-1}(A),\mathbb{R}^n)\right\}.\end{aligned}$$

Let then $\overline{Q} \subset \tilde{Q}$ be a cube such that $|\tilde{Q}\setminus\overline{Q}| \leq \eta$. Using (6) and (25), for t big enough, there holds

$$\nu_{\Phi(\omega'),\xi}(\omega)(t\tilde{Q}) \leq \nu_{L,\xi}(\omega)(t\overline{Q}) + t^n\eta C(1+|\xi|^p).$$

Thus,

$$\limsup_{t\to\infty} \frac{\nu_{\Phi(\omega'),\xi}(\omega)(t\tilde{Q})}{|t\tilde{Q}|} \leq (1-\eta)\mathcal{V}\phi(\xi) + \eta C(1+|\xi|^p).$$

Conversely, for $\underline{Q}$ a cube containing $\tilde{Q}$ such that $|\underline{Q}\setminus\tilde{Q}| \leq \eta$, and for t big enough, there holds

$$\nu_{L,\xi}(\omega)(t\underline{Q}) \leq \nu_{\Phi(\omega'),\xi}(\omega)(t\tilde{Q}) + t^n\eta C(1+|\xi|^p).$$

We then obtain

$$\mathcal{V}\phi(\xi) \leq (1-\eta)\liminf_{t\to\infty} \frac{\nu_{\Phi(\omega'),\xi}(\omega)(t\tilde{Q})}{|t\tilde{Q}|} + \eta C(1+|\xi|^p).$$

Therefore, almost surely (in ω and ω'),

$$\lim_{t\to\infty} \frac{\nu_{\Phi(\omega'),\xi}(\omega)(t\tilde{Q})}{|t\tilde{Q}|} = \mathcal{V}\phi(\xi) \tag{26}$$

Let us now go back to I_ϵ. Let $\omega' \in \Omega$ be fixed and consider $\Psi(x) = \Phi(x,\omega)$. For all cube $\tilde{Q}$ of $\mathbb{R}^n$ we then have

$$\begin{aligned} m_\epsilon(\omega,\Psi)(\tilde{Q},\xi) &= \inf\left\{\int_{\tilde{Q}} W\left(\Phi^{-1}\left(\frac{y}{\epsilon},\omega\right), \xi + \nabla u\right) dy, u \in W_0^{1,p}(\tilde{Q},\mathbb{R}^d)\right\} \\ &= \epsilon^n \inf\left\{\int_{\Psi^{-1}\left(\frac{1}{\epsilon}\tilde{Q}\right)} W\left(y, \xi + \left(\nabla\Phi\left(y,\omega\right)\right)^{-1}\nabla\tilde{u}\right)\det\left(\nabla\Phi\left(y,\omega\right)\right) dy,\right. \\ &\qquad\qquad \left.\tilde{u} \in W_0^{1,p}\left(\Psi\left(\frac{1}{\epsilon}\tilde{Q}\right),\mathbb{R}^d\right)\right\}, \end{aligned} \tag{27}$$

On the one hand, due to Lemma 1 and Theorem 5, almost surely, there exists an extraction function α and a standard energy density $\tilde{W}$, such that for every cube $\tilde{Q}$ in $\mathbb{R}^n$,

$$\lim_{k\to\infty} m_{\epsilon_{\alpha(k)}}(\omega,\Psi)(\tilde{Q},\xi) = \inf\left\{\int_{\tilde{Q}} \tilde{W}\left(x,\omega,\xi+\nabla u\right), u \in W_0^{1,p}(\tilde{Q})\right\}, \tag{28}$$

On the other hand,

$$\lim_{t\to\infty} \frac{1}{|t\tilde{Q}|} \int_{\Psi^{-1}(t\tilde{Q})} dy = \mathcal{V} = \det\left(\mathbb{E}\left(\int_Q \nabla\Phi(y,\cdot)dy\right)\right)^{-1}.$$

Thus, (26) implies

$$\lim_{\epsilon\to 0} \frac{1}{|\tilde{Q}|} m_\epsilon(\omega,\Psi)(\tilde{Q},\xi) = \det\left(\mathbb{E}\left(\int_Q \nabla\Phi(y,\cdot)dy\right)\right)^{-1} \phi(\xi) = W^*(\xi). \tag{29}$$

In particular for every Lebesgue point of $\tilde{W}(\cdot,\omega,\xi)$, and therefore almost everywhere, (28) and (29) show that

$$\tilde{W}(x,\omega,\xi) = W^*(\xi),$$

which holds for almost all $x \in \mathbb{R}^n$, all $\xi \in \mathbb{R}^{n\times d}$ and for $\mathbb{P}$-almost every $\omega \in \Omega$.

For the only possible Γ-limit is F^*, the whole family of integral functionals Γ-converges to F^* almost surely, which concludes the proof of Theorem 3.

Remark 1 *The proof of Theorem 4 is the same as the proof above up to replacing $W(x,\xi)$ by $W(x,\omega,\xi)$ and noticing that the stationarity of G holds due to the stationarity of $\nabla\Phi$ and to Hypotheses 2.*

Let us now identify some abstract properties that ensure the homogenization result. We will denote by admissible diffeomorphism a diffeomorphism Ψ satisfying (3) and (4) and the average property: there exists $\mathcal{V}_\Psi \in \mathbb{R}$ such that

$$\mathcal{V}_\Psi = \lim_{N\to\infty} \frac{1}{N^n} \int_{\Psi^{-1}(Q_N)} dz.$$

Remark 2 *Let $W(x,\omega,\xi)$ be a stochastic family of standard energy densities and Φ be a random family of diffeomorphisms satisfying (3) and (4) and for which there exists $\mathcal{V} \in \mathbb{R}$ such that*

$$\lim_{\epsilon\to 0} \frac{1}{|\frac{1}{\epsilon}A|} \int_{\Phi^{-1}(\frac{1}{\epsilon}|A|,\omega)} dx = \mathcal{V}$$

for all $A \in \mathcal{A}$, $\mathbb{P}$-almost surely. Let denote by $\mathcal{G} : \mathcal{A} \times \mathbb{R}^{n\times d} \times \Omega$,

$$\mathcal{G} : (A,\xi,\omega) \mapsto \inf\left\{\int_A W\Big(y,\omega,\xi + (\nabla\Phi\,(y,\omega))^{-1}\,\nabla v\Big) \det(\nabla\Phi(y,\omega)), v \in W_0^{1,p}(A)\right\}.$$

If for all $\xi \in \mathbb{R}^{n\times d}$, there exists $\phi(\xi) \in \mathbb{R}$ such that for all $A \in \mathcal{A}$ and for all admissible diffeomorphism Ψ, almost surely,

$$\lim_{\epsilon\to 0} \frac{\epsilon^n}{|A|} \mathcal{G}(\Psi^{-1}(\frac{1}{\epsilon}A),\xi,\omega) = \phi(\xi)\mathcal{V}_\Psi, \tag{30}$$

then, for all $A \in \mathcal{A}$, the integral functional $F_\epsilon : (A,v,\omega) \mapsto \int_A W(\Phi^{-1}(\frac{x}{\epsilon}),\omega,\nabla v)$ $\Gamma(L^p)$-converges to $F^* : (A,v) \mapsto \int_A \mathcal{V}\phi^*(\nabla v)$ on $W^{1,p}(A)$, $\mathbb{P}$-almost surely.

Property (30) is an average property. Remark 2 roughly tells that if $\nabla\Phi$ and W are such that $\mathcal{G}(t\tilde{Q},\xi,\omega)$ admits a deterministic average for all cube $\tilde{Q}$, then the homogenization property holds.

3.3 Proof of Lemma 2

The proof of Lemma 2 is a more abstract proof using the definitions and properties of Γ-convergence. It is however based on the same arguments as in Subsection 3.2. Due to the growth condition (6), we will assume the energy density to be non negative.

Let $A \in \mathcal{A}$. Let $u_k \to u \in L^p(A)$ be such that the energy is bounded and denote by

$$\begin{array}{lcll}\bar{u}_k & : & \epsilon_k\Phi^{-1}(\frac{A}{\epsilon_k},\omega) & \to \mathbb{R}^d, \\ & & y & \mapsto u_k(\epsilon_k\Phi(\frac{y}{\epsilon_k},\omega)).\end{array}$$

In what follows, let ω be fixed. The results below hold almost surely.

Let us recall that, as proved in Subsection 3.2, for all $B \in \mathcal{A}$ such that $B \subset\subset L^{-1}(A)$ and for k big enough, $B \subset \epsilon_k\Phi^{-1}(\frac{A}{\epsilon_k},\omega)$. Therefore $\bar{u}_k \in L^p(B)$. We claim that

$$\bar{u}_k \to \bar{u} = u \circ L \qquad \text{in } L^p(B).$$

Actually,

$$\begin{array}{ll} & \int_B |\bar{u}_k(y) - \bar{u}(y)|^p \\ \leq & C_p\int_B |\bar{u}_k(y) - u(\epsilon_k\Phi(\frac{y}{\epsilon_k},\omega))|^p + C_p\int_B |u(\epsilon_k\Phi(\frac{y}{\epsilon_k},\omega)) - \bar{u}(y)|^p.\end{array} \tag{31}$$

After a change of variable, and noticing that $\epsilon_k\Phi_k(\frac{B}{\epsilon_k}) \subset A$, the uniform boundness of the Jacobian and the convergence of u_k to u in $L^p(A)$ shows that the first term of (31) vanishes as $k \to \infty$. Let us now deal with the second term. As $u \in L^p(A)$,

$$\forall\eta > 0, \exists\beta_\eta > 0, \forall E \subset A, |E| \leq \beta_\eta \implies \int_E |u|^p \leq \eta.$$

Let $\eta > 0$, and $E \subset B$ be such that $|E| \leq \frac{\beta_\eta}{M^n}$. Then, due to (4) and (19),

$$\int_E |u(\epsilon_k\Phi(\frac{y}{\epsilon_k},\omega))|^p \leq C\int_{\epsilon_k\Phi(\frac{E}{\epsilon_k},\omega)} |u|^p \leq C\eta,$$

where C does not depend on k and E. The sequence $\{u(\epsilon_k\Phi(\frac{\cdot}{\epsilon_k},\omega))\}_k$ is thus p-equi-integrable. For every Lebesgue point $x = L(y) \in A$ of u, $u(\epsilon_k\Phi(\frac{y}{\epsilon_k},\omega)) \to u(x)$. Therefore, $|u(\epsilon_k\Phi(\frac{y}{\epsilon_k},\omega)) - \bar{u}(y)|^p \to 0$ almost everywhere on B. The Egorov theorem then implies the quasi-uniform convergence of $|u(\epsilon_k\Phi(\frac{y}{\epsilon_k},\omega)) - \bar{u}(y)|^p$ to zero on B. The L^1-norm of $|u(\epsilon_k\Phi(\frac{y}{\epsilon_k},\omega)) - \bar{u}(y)|^p$ on the subset where the convergence is not uniform is then controlled using the p-equi-integrability of the sequence (see [20, Appendix] for a similar proof), which shows that the second term of (31) vanishes as $k \to 0$ and concludes the proof of the claim.

We are now in position to prove the Γ-liminf inequality. For k big enough,

$$\begin{array}{rcl}\liminf_k I_{\epsilon_k}(u_k,A) & = & \liminf_k \int_{\epsilon_k\Phi^{-1}(\frac{A}{\epsilon_k},\omega)} W(\frac{y}{\epsilon_k},\omega,(\nabla\Phi(\frac{y}{\epsilon_k},\omega))^{-1}\nabla\bar{u}_k)\det(\nabla\Phi(\frac{y}{\epsilon_k},\omega))dy \\ & \geq & \liminf_k \int_B W(\frac{y}{\epsilon_k},\omega,(\nabla\Phi(\frac{y}{\epsilon_k},\omega))^{-1}\nabla\bar{u}_k)\det(\nabla\Phi(\frac{y}{\epsilon_k},\omega))dy \\ & \geq & \int_B \phi^*(\nabla\bar{u}) \\ & \geq & \int_{L^{-1}(A)} \phi^*(\nabla\bar{u}) - C\|\bar{u}\|^p_{W^{1,p}(L^{-1}(A)\setminus B)} \\ & = & \int_A \mathcal{V}\phi^*(\nabla u) - C\|\bar{u}\|^p_{W^{1,p}(L^{-1}(A)\setminus B)}.\end{array}$$

As this inequality holds for all $B \subset\subset L^{-1}(A)$, the liminf inequality is proved.

We proceed the same way for the Γ-limsup. Let us consider $D \in \mathcal{A}$, such that $L^{-1}(A) \subset\subset D$. Let us extend $\bar{u}$ on $W^{1,p}(D)$ and consider a recovery sequence $\bar{v}_k \to \bar{u}$ in $L^p(D)$ for J_ϵ. We set $v_k : A \to \mathbb{R}^d, y \mapsto \bar{v}_k(\epsilon_k\Phi^{-1}(\frac{y}{\epsilon_k},\omega))$, which is well-defined. Proceeding as above, there holds that $v_k \to u$ in $L^p(A)$. We then have

$$\begin{array}{rcl}\limsup_k I_{\epsilon_k}(v_k,A) & = & \limsup_k \int_{\epsilon_k\Phi^{-1}(\frac{A}{\epsilon_k},\omega)} W(\frac{y}{\epsilon_k},\omega,(\nabla\Phi(\frac{y}{\epsilon_k},\omega))^{-1}\nabla\bar{v}_k)\det(\nabla\Phi(\frac{y}{\epsilon_k},\omega))dy \\ & \leq & \limsup_k \int_D W(\frac{y}{\epsilon_k},\omega,(\nabla\Phi(\frac{y}{\epsilon_k},\omega))^{-1}\nabla\bar{v}_k)\det(\nabla\Phi(\frac{y}{\epsilon_k},\omega))dy \\ & \leq & \int_D \phi^*(\nabla\bar{u}) \\ & \leq & \int_{L^{-1}(A)} \phi^*(\nabla\bar{u}) + C\|\bar{u}\|^p_{W^{1,p}(D\setminus L^{-1}(A))} \\ & \leq & \int_A \mathcal{V}\phi^*(\nabla u) + C\|\bar{u}\|^p_{W^{1,p}(D\setminus L^{-1}(A))}.\end{array}$$

Noticing that for all $D' \in \mathcal{A}$ such that $D' \subset D$, $\bar{v}_k$ is still a recovery sequence on D', we deduce that the limsup inequality holds.

As a consequence, I_ϵ $\Gamma(L^p)$-converges to $I^* = \mathcal{V}J^*$.

3.4 Ergodicity for the continuous group of translations

In the case of a continuous group of translations, any stationary subadditive process μ is obviously also stationary in the discrete sense (or periodic in law). Thus, [8, Theorem 2.7] implies the existence of a full measure borelian subset Ω' of Ω and of a measurable function $\phi : \Omega' \to \mathbb{R}$ such that

$$\lim_{N\to\infty} \frac{\mu(\omega)(\tilde{Q}_N)}{|\tilde{Q}_N|} = \phi(\omega)$$

for every $\omega \in \Omega'$ and for any sequence of cubes of side N with vertices in $\mathbb{Z}^n$. If the subadditive process is sublinear, then, by an easy approximation argument, the conclusion holds for any family of cubes in $\mathbb{R}^n$. Using the stationarity of the process, we then have that for every $\omega \in \Omega'$ and for all $z \in \mathbb{R}^d$, $\phi(\tau_z\omega) = \phi(\omega)$. Due to the ergodicity of the translation group and to the uniform boundness of ϕ, the latter equality implies that $\phi(\cdot)$ is constant with probability one on Ω', which yields the conclusion.

Acknowledgement.

The author is grateful to Xavier Blanc and Claude Le Bris for their valuable remarks and comments on the manuscript.

References

[1] X. Blanc, C. Le Bris, and P.L. Lions. Une variante de la théorie de l'homogénéisation stochastique des opérateurs elliptiques. *C. R. Acad. Sci. Paris, Série I*, 343:717–724, 2006.

[2] X. Blanc, C. Le Bris, and P.L. Lions. Stochastic homogenization and random lattices. *J. Math. Pures Appl.*, 88(1):34–63, 2007.

[3] X. Blanc, C. Le Bris, and P.L. Lions. Du discret au continu pour des réseaux aléatoires d'atomes. *C. R. Acad. Sci. Paris, Série I*, 342:627–633, 2006.

[4] X. Blanc, C. Le Bris, and P.L. Lions. The energy of some microscopic stochastic lattices. *Arch. Rational Mech. Anal.*, 184(2):303–339, 2007.

[5] G. Dal Maso and L. Modica. Nonlinear stochastic homogenization and ergodic theory. *J. Reine Angew. Math.*, 368:28–42, 1986.

[6] R. Alicandro, M. Cicalese, and A. Gloria. Mathematical derivation of a rubber-like stored energy functional. *C. R. Acad. Sci. Paris, Série I*, in press.

[7] R. Alicandro, M. Cicalese, and A. Gloria. Integral representation results for energies defined on stochastic lattices and application to nonlinear elasticity. In preparation.

[8] M.A. Akcoglu and U. Krengel. Ergodic theorems for superadditive processes. *J. reine angew. Math.*, 323:53–67, 1981.

[9] V.V. Jikov, S.M. Kozlov, and O.A. Oleinik. *Homogenization of Differential Operators and Integral Functionals*. Springer-Verlag, Berlin, 1994.

[10] U. Krengel. *Ergodic theorems*, volume 6 of *de Gruyter Studies in Mathematics*. De Gruyter, 1985.

[11] A. Braides. *Γ-convergence for beginners*, volume 22 of *Oxford Lecture Series in Mathematics and Its Applications*. Oxford University Press, 2002.

[12] A. Braides. A handbook of Γ-convergence. volume 3 of *Handbook of Differential Equations: Stationary Partial Differential Equations*, pages 101–213. Elsevier, Amsterdam, 2006.

[13] G. Dal Maso. *An Introduction to Γ-Convergence*. Birkhäuser Boston, Boston, MA, 1993.

[14] A. Braides. Homogenization of some almost periodic functionals. *Rend. Accad. Naz. Sci. XL*, 103:261–281, 1985.

[15] A. Braides and A. Defranceschi. *Homogenization of Multiple Integrals*, volume 12 of *Oxford Lecture Series in Mathematics and Its Applications*. Oxford University Press, 1998.

[16] S. Müller. Homogenization of nonconvex integral functionals and cellular elastic materials. *Arch. Rat. Anal. Mech.*, 99:189–212, 1987.

[17] C. Licht and G. Michaille. Global-local subadditive ergodic theorems and application to homogenization in elasticity. *An. math. Blaise Pascal*, 9(1):21–62, 2002.

[18] P. Marcellini. Periodic solutions and homogenization of nonlinear variational problems. *Annali Mat. Pura Appl.*, 117:139–152, 1978.

[19] D. Ruelle. *Statistical mechanics. Rigorous results.* World Scientific Publishing Co., Inc., River Edge, NJ; Imperial College Press, London, 1999. Reprint of the 1989 edition.

[20] A. Gloria. An analytical framework for the numerical homogenization of monotone elliptic operators and quasiconvex energies. *Multiscale Model. Simul.*, 5(3):996–1043, 2006.

Unité de recherche INRIA Rocquencourt
Domaine de Voluceau - Rocquencourt - BP 105 - 78153 Le Chesnay Cedex (France)

Unité de recherche INRIA Futurs : Parc Club Orsay Université - ZAC des Vignes
4, rue Jacques Monod - 91893 ORSAY Cedex (France)
Unité de recherche INRIA Lorraine : LORIA, Technopôle de Nancy-Brabois - Campus scientifique
615, rue du Jardin Botanique - BP 101 - 54602 Villers-lès-Nancy Cedex (France)
Unité de recherche INRIA Rennes : IRISA, Campus universitaire de Beaulieu - 35042 Rennes Cedex (France)
Unité de recherche INRIA Rhône-Alpes : 655, avenue de l'Europe - 38334 Montbonnot Saint-Ismier (France)
Unité de recherche INRIA Sophia Antipolis : 2004, route des Lucioles - BP 93 - 06902 Sophia Antipolis Cedex (France)

Éditeur
INRIA - Domaine de Voluceau - Rocquencourt, BP 105 - 78153 Le Chesnay Cedex (France)
http://www.inria.fr
ISSN 0249-6399